Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
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doi:10.55730/1300-0098.

Curves as slant submanifolds of an almost product Riemannian manifold

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Received: 18.09.2023 • **Accepted/Published Online:** 25.03.2024 • **Final Version:** ..202

Abstract: In this paper, we show that in an almost product manifold there exist curves that are slant submanifolds. We characterize these curves and study them in two and three-dimensional locally product manifolds. Finally, we construct curves in a hypersurface of a Kaehler manifold.

Key words: Slant submanifolds, curve, almost product structure.

1. Introduction

In [8], B. Sahin introduced the notion of a slant submanifold of an almost product manifold. He noted that although in the almost complex case a slant submanifold is even dimensional, and in the almost contact case it is odd-dimensional, in the almost product case the slant submanifold could be both even or odd-dimensional. He presented interesting examples: surfaces in $R^2 \times R^2$ and R^4 and a 3-dimensional example in $R^4 \times R^3$.

Different classes of Riemannian almost product manifolds have been studied in the literature (bundle-like metrics, conformal foliations, minimal and geodesic plane fields, minimal foliations, totally geodesic foliations...). In [7] A.M. Naveira gave a classification of almost product manifolds in 36 classes.

Several authors have continued the study of slant submanifolds of an almost product manifold [1, 9–12] and also slant submersions [4]. Slant submanifolds were first defined by B.-Y. Chen in an almost complex manifold, [3], and later studied in an almost contact environment, [2, 5]. In both cases, the existence of slant curves has no sense, as any curve is antiinvariant. Surprisingly, in the almost product manifold case, it is possible the existence of curves that are slant submanifolds. We avoid calling them slant curves, as that name has been used in almost metric contact geometry for curves whose tangent vector field forms a constant angle with the structure vector field ξ.

The study of different types of curves (geodesic, Legendrian, slant, magnetic...) is an interesting field of research. This notion of curves that are slant submanifolds both introduces a new type of curves susceptible to be studied in different ambient spaces and reveals a particular example of slant submanifolds that have never been considered.

The paper is organized as follows: after a first section with the preliminaries, where we remember all the results from [8] about slant submanifolds of almost product manifolds, we give a characterization of the curves

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2010 *AMS Mathematics Subject Classification:* 53C40, 53C25, 53C15

under study and present some examples. In Section 4, curves in a locally product, with dimension 2 or 3, are studied obtaining their curvature and torsion. Finally, we look for new nontrivial examples and constructions: in the fifth section, we construct different almost product structures over the Euclidean plane, adapted to the curve for it to be a slant submanifold, and in the last section curves in hypersurfaces of a Kaehlerian manifold are studied.

2. Preliminaries

In this section, we recall some basic formulas about almost product manifolds.

A Riemannian manifold $(\tilde{M}, g)$ is called an *almost product Riemannian manifold*, [13], if there exists a $(1,1)$ tensor field F such that

$$F^2 = I, \quad g(FX, FY) = g(X, Y), \tag{2.1}$$

for any vector fields X, Y on M, where I denotes the identity transformation, $F \neq \pm I$. And it is called a *locally product Riemannian manifold* if $(\nabla_X F)Y = 0$, for any X, Y.

Let M be a Riemannian manifold isometrically immersed in an almost product Riemannian manifold $\tilde{M}$. For any tangent vector field X in M we write

$$FX = TX + NX, \tag{2.2}$$

where TX is the tangential component of FX and NX is the normal one. For any normal vector field V

$$FV = tV + nV, \tag{2.3}$$

where tV and nV denote the tangent and normal parts of FV.

For each nonzero vector field X tangent to M at x, the angle $\theta(X)$, $0 \leq \theta(X) \leq \pi/2$ between FX and T_xM is called the *Wirtinger angle*. If this angle $\theta(X)$ is constant, independently of the choice of $x \in M$ and $X \in T_xM$, then the submanifold is called *slant submanifold* (B. Sahin [8]). F-invariant and F-antiinvariant submanifolds are special cases of slant submanifolds for $\theta = 0$ and $\theta = \pi/2$, respectively.

Theorem 2.1 *[8] Let M be a submanifold of an almost product Riemannian manifold $\tilde{M}$. Then, M is a slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$T^2 = \lambda I. \tag{2.4}$$

Note that, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

If M is a slant submanifold, from (2.1) using (2.2,2.3,2.4) it holds

$$\begin{aligned} tNX &= sin^2\theta X, \\ NTX + nNX &= 0. \end{aligned} \tag{2.5}$$

Remark 2.2 *In an almost complex manifold (N^{2n}, J, g), JX is orthogonal to X for any tangent vector field X. The same happens for $X, \varphi X$ in an almost contact manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$. Therefore, someone "living" on a curve would have no notion of the structure.*

Now, in an almost product manifold, the situation is completely different: X and FX are not necessarily orthogonal. Someone "living" on a curve would perceive some information of the almost contact structure: he takes the tangent vector field $\mathbf{t}$ and recover $F\mathbf{t}$ that, generally, will be shorter. If that curve is a slant submanifold, the perception of the structure, the loose of length, is the same along the curve.

3. Curves that are slant submanifolds. Examples

As we have said in the Introduction, the definition of slant submanifolds of an almost product manifold given by B. Sahin, [8], admits both even and odd dimensional submanifolds. Moreover, in opposite to both the almost complex and almost contact environment, in an almost product manifold, it is possible for a curve to be a slant submanifold. This is because from (2.1) it is possible to consider a vector field X satisfying $g(X, FX) \neq 0$.

From Theorem 2.1 we can give an easier characterization of these curves that are slant submanifolds.

Proposition 3.1 *Let γ be a smooth curve of an almost product Riemannian manifold $\tilde{M}$. Then, γ is a slant submanifold if and only if $T\dot{\gamma} = \pm\cos\theta\dot{\gamma}$*

Proof It follows directly from the definition of slant submanifold by taking into account the decomposition $F\dot{\gamma} = T\dot{\gamma} + N\dot{\gamma}$. If the curve is a slant submanifold, taking into account the definition of slant angle, the angle between FX and the tangent space of the submanifold, it is directly deduced that $T\dot{\gamma} = \pm\cos\theta\dot{\gamma}$. □

From now on, we will use this characterization with $TX = \cos\theta X$, but $\theta \in [0, \pi]$.

Proposition 3.2 *Let γ be a smooth curve in an almost product Riemannian surface $\tilde{M}$. Let $\mathbf{t}$ and $\mathbf{n}$ its tangent and normal vector fields. Then*

i) if γ is invariant, $F\mathbf{t} = \pm\mathbf{t}$ and $F\mathbf{n} = \mp\mathbf{n}$,

ii) if γ is antiinvariant, $F\mathbf{t} = \pm\mathbf{n}$ and $F\mathbf{n} = \pm\mathbf{t}$.

Proof

i) If γ is invariant, $F\mathbf{t} = \pm\mathbf{t}$,

$$g(F\mathbf{n}, \mathbf{t}) = g(\mathbf{n}, F\mathbf{t}) = 0,$$

and $F\mathbf{n}$ is orthogonal to $\mathbf{t}$. As F preserves the length, it must be $\mathbf{n}$ or $-\mathbf{n}$. And, as F is not $\pm Id$, it has the opposite sign than $F\mathbf{t}$, that is $F\mathbf{n} = \mp\mathbf{n}$.

ii) If γ is antiinvariant, $F\mathbf{t} = \pm\mathbf{n}$. And $F\mathbf{n} = \pm F^2\mathbf{t} = \pm\mathbf{t}$.

□

Let us show some examples:

Example 3.3 *Consider $\mathbb{R}\times\mathbb{R}$ with coordinates x, y and product structure*

$$F\frac{\partial}{\partial x} = \frac{\partial}{\partial x}, \quad F\frac{\partial}{\partial y} = -\frac{\partial}{\partial y}. \tag{3.1}$$

For any $0 \leq a \leq \frac{\pi}{2}$, the straight line given by

$$\alpha(u) = (u\cos a, u\sin a)$$

is a slant submanifold with slant angle $\theta = 2a$.

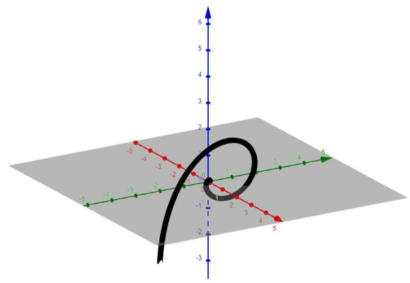

Figure 1. A slant curve from Example 3.6 with $k = 1/2$.

Example 3.4 *Consider* $\mathbb{R} \times \mathbb{R}$ *with coordinates* x, y *and product structure*

$$F\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad F\frac{\partial}{\partial y} = \frac{\partial}{\partial x}. \tag{3.2}$$

For any $0 \leq b \leq \frac{\pi}{4}$, *the straight line given by*

$$\beta(u) = (u \cos b, u \sin b)$$

is a slant submanifold. Actually, $\dot{\beta} = (\cos b, \sin b)$ *and the Wirtinger angle is given by* $\cos\theta = g(\dot{\beta}, F\dot{\beta}) = 2\sin b \cos b = \cos(\pi/2 - 2b)$, *so the slant angle is* $\theta = \pi/2 - 2b$.

For the next examples, consider $\mathbb{R} \times \mathbb{R}^2$ with coordinates x, y, z and product structure

$$F\frac{\partial}{\partial x} = \frac{\partial}{\partial x}, \quad F\frac{\partial}{\partial y} = -\frac{\partial}{\partial y}, \quad F\frac{\partial}{\partial z} = -\frac{\partial}{\partial z}. \tag{3.3}$$

Example 3.5 *For any* k, *the helix given by*

$$\alpha(u) = (u, -k \sin u, k \cos u)$$

is a slant submanifold with slant angle $\theta = \cos^{-1}\left(\frac{1-k^2}{1+k^2}\right)$.

Example 3.6 *For any* k, *the curve given by*

$$\beta(u) = (e^{ku}, e^{ku}\cos u, e^{ku}\sin u)$$

is a slant submanifold. Actually, $\dot{\beta} = \frac{1}{\sqrt{2k^2+1}}(k, k\cos u - \sin u, k \sin u + \cos u)$ *and the Wirtinger angle is given by* $\cos\theta = g(\dot{\beta}, F\dot{\beta}) = \frac{-1}{2k^2+1}$ *so the slant angle is* $\theta = \cos^{-1}\left(\frac{-1}{2k^2+1}\right)$.

A picture of this curve for $k = 1/2$ can be seen in Figure 1.

Finally, consider $\mathbb{R} \times \mathbb{R}^2$ with coordinates x, y, z and product structure

$$F\frac{\partial}{\partial x} = \frac{\partial}{\partial x}, \quad F\frac{\partial}{\partial y} = \frac{\partial}{\partial z}, \quad F\frac{\partial}{\partial z} = \frac{\partial}{\partial y}. \tag{3.4}$$

Example 3.7 *For any a, b with $a^2 + b^2 \leq 1$, the straight line given by*

$$\alpha(u) = (u, au, bu)$$

is a slant submanifold with slant angle $\theta = \cos^{-1}\left(\dfrac{1 - a^2 - b^2}{1 + a^2 + b^2}\right)$.

3.1. Curves in $\mathbb{R} \times \mathbb{R}$

At Examples 3.3, 3.4 we present straight lines as slant curves. Now we confirm that, with such a structure, the only slant curves are straight lines.

Consider $\gamma(u) = (x(u), y(u))$, a unit speed curve, and let $\mathbf{t} = (x', y')$ be the tangent vector field. From (3.1), $F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $F\mathbf{t} = (x', -y')$

Imposing, that $|\mathbf{t}| = 1$ and that $\mathbf{t}$ and $F\mathbf{t}$ makes a constant angle θ, we get the following differential system:

$$\left\{ \begin{array}{rcl} x'^2 + y'^2 & = & 1, \\ x'^2 - y'^2 & = & \cos\theta. \end{array} \right.$$

Taking initial conditions $\gamma(0) = (0, 0)$, the solution is

$$\gamma(u) = \left(\sqrt{\frac{1 + \cos\theta}{2}}u, \sqrt{\frac{1 - \cos\theta}{2}}u \right),$$

that is and straight line.

For the second structure (3.2), $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so $F\mathbf{t} = (y', x')$ and the system is:

$$\left\{ \begin{array}{rcl} x'^2 + y'^2 & = & 1, \\ 2x'y' & = & \cos\theta, \\ x(0) & = & 0, \\ y(0) & = & 0, \end{array} \right.$$

with solution

$$\gamma(u) = \left(\frac{\sqrt{1 + \cos\theta} \pm \sqrt{1 - \cos\theta}}{2}u, \frac{\sqrt{1 + \cos\theta} \mp \sqrt{1 - \cos\theta}}{2}u \right),$$

again a straight line.

In general, at the Euclidean plane, a constant almost product structure would be given by a matrix $F = \begin{pmatrix} \cos A & \sin A \\ \sin A & -\cos A \end{pmatrix}$ the system is:

$$\left\{ \begin{array}{rcl} x'^2 + y'^2 & = & 1, \\ \cos A(x'^2 - y'^2) + 2\sin A x'y' & = & \cos\theta, \\ x(0) & = & 0, \\ y(0) & = & 0, \end{array} \right.$$

for certain constant A. This can be solved using any computation program and the solution is a straight line.

Obviously, if the almost product structure is not constant, it could exist nonstraight slant curves.

3.2. Curves in $\mathbb{R} \times \mathbb{R}^2$

Let be a unit speed curve $\gamma(u) = (x, y, z)$, with tangent vector field $\mathbf{t} = (x', y', z')$ in $\mathbb{R}\times\mathbb{R}^2$. First, taking the structure (3.3), $F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $F\mathbf{t} = (x', -y', -z')$. Imposing the unit and the slant conditions, we get the following system

$$\left\{ \begin{array}{ccc} x'^2 + y'^2 + z'^2 & = & 1, \\ x'^2 - y'^2 - z'^2 & = & \cos\theta, \end{array} \right.$$

that is equivalent to

$$\left\{ \begin{array}{ccc} x'^2 & = & \dfrac{1+\cos\theta}{2} \\ y'^2 + z'^2 & = & \dfrac{1-\cos\theta}{2}. \end{array} \right.$$

A particular solution could be obtained as

$$\left\{ \begin{array}{ccl} x' & = & \sqrt{\dfrac{1+\cos\theta}{2}} \\ y' & = & \sqrt{\dfrac{1-\cos\theta}{2}}\sin u \\ z' & = & \sqrt{\dfrac{1-\cos\theta}{2}}\cos u. \end{array} \right.$$

and hence if we take initial conditions $\gamma(0) = (0, 0, 0)$, the solution is

$$\gamma(u) = \left(\sqrt{\frac{1+\cos\theta}{2}}u, -\sqrt{\frac{1-\cos\theta}{2}}(\cos u - 1), \sqrt{\frac{1-\cos\theta}{2}}\sin u \right)$$

Note, we do not get exactly Example 3.7 as this is a particular solution.

And finally, for structure (3.4), $F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $F\mathbf{t} = (x', z', y')$, and the system is

$$\left\{ \begin{array}{ccc} x'^2 + y'^2 + z'^2 & = & 1, \\ x'^2 + 2y'z' & = & \cos\theta, \end{array} \right.$$

adding and subtracting both equations we get

$$\left\{ \begin{array}{ccc} 2x'^2 + (y' + z')^2 & = & 1 + \cos\theta \\ (y' - z')^2 & = & 1 - \cos\theta. \end{array} \right.$$

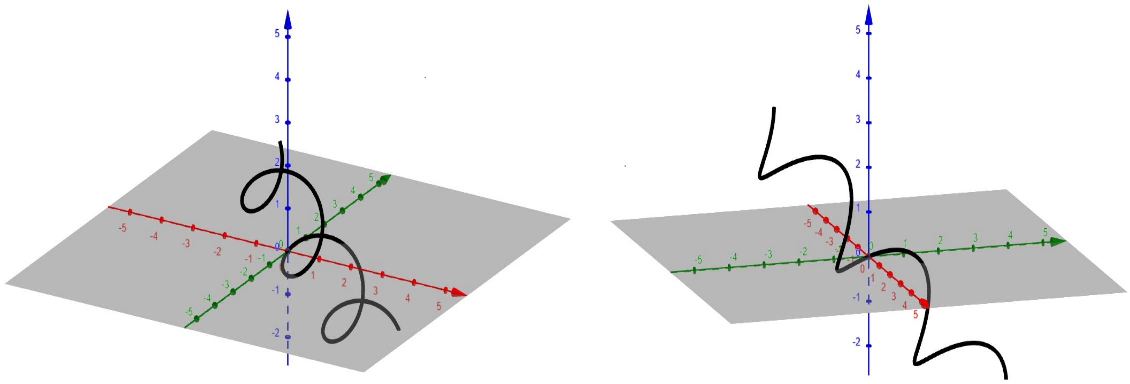

Figure 2. The slant curve in $\mathbb{R}\times\mathbb{R}^2$ given by (3.5) with $\theta=\pi/3$.

A particular solution could be obtained as

$$\begin{cases} 2x'^2 &= (1+\cos\theta)\sin^2 u\\ (y'+z')^2 &= (1+\cos\theta)\cos^2 u\\ (y'-z')^2 &= 1-\cos\theta, \end{cases}$$

and hence

$$\begin{cases} x' &= \sqrt{\dfrac{1+\cos\theta}{2}}\sin u\\ y' &= \dfrac{\sqrt{1+\cos\theta}}{2}\cos u+\dfrac{\sqrt{1-\cos\theta}}{2}\\ z' &= \dfrac{\sqrt{1+\cos\theta}}{2}\cos u-\dfrac{\sqrt{1-\cos\theta}}{2}. \end{cases}$$

If we take initial conditions $\gamma(0)=(0,0,0)$, the solution is

$$\gamma(u)=\left(-\sqrt{\frac{1+\cos\theta}{2}}(\cos u-1),\frac{\sqrt{1+\cos\theta}}{2}\sin u+\frac{\sqrt{1-\cos\theta}}{2}u,\frac{\sqrt{1+\cos\theta}}{2}\sin u-\frac{\sqrt{1-\cos\theta}}{2}u\right). \tag{3.5}$$

A picture of this curve for $\theta=\pi/3$ can be seen in Figure 2.

4. Slant curves of locally product Riemannian manifolds

If the structure over the manifold is not just almost product but a locally product one, we will be able to characterize curves which are slant submanifolds both in the 2 and 3-dimensional cases.

Theorem 4.1 *Let γ be a smooth curve parametrized by its arc length of a 2-dimensional locally product Riemannian manifold, $\widetilde{M}^2$. If γ is a slant submanifold, then it is invariant or a geodesic.*

Proof Consider $\{\mathbf{t},\mathbf{n}\}$ the Frenet reference on the curve, and write $\nabla_{\mathbf{t}}\mathbf{t}=\kappa\mathbf{n}$, with κ the curvature. Since γ is a slant curve, the angle between $\mathbf{t}$ and $F\mathbf{t}$ is constant. Given that $\widetilde{M}$ is a locally product Riemannian manifold, $\nabla F=0$. Then, differentiating

$$\mathbf{t}g(\mathbf{t},F\mathbf{t})=g(\nabla_{\mathbf{t}}\mathbf{t},F\mathbf{t})+g(\mathbf{t},F\nabla_{\mathbf{t}}\mathbf{t})=2g(\nabla_{\mathbf{t}}\mathbf{t},F\mathbf{t}),$$

which implies:

$$0 = g(\kappa\mathbf{n}, F\mathbf{t}). \tag{4.1}$$

Then, either $\kappa = 0$ and the curve is a geodesic or $F\mathbf{t}$ is orthogonal to $\mathbf{n}$, in this case $F\mathbf{t} = \pm\mathbf{t}$ and γ is an invariant curve. □

For 3-dimensional locally product manifolds, we present the following

Theorem 4.2 *Let γ be a smooth curve parametrized by its arc length of a 3-dimensional locally product Riemannian manifold, $\tilde{M}^3$. If γ is a slant submanifold, then it is invariant, geodesic, or with curvature and torsion satisfying the following relation*

$$\tau = \pm\frac{\kappa(1+\cos\theta)}{\sin\theta}, \tag{4.2}$$

where θ is the slant angle.

Proof The Frenet-Serret formula is given by

$$\begin{array}{lcl} \nabla_{\mathbf{t}}\mathbf{t} & = & \kappa\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} & = & -\kappa\mathbf{t} \quad +\tau\mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} & = & -\tau\mathbf{n}. \end{array} \tag{4.3}$$

Since γ is a slant curve, the angle between $\mathbf{t}$ and $F\mathbf{t}$ is constant. Given that $\tilde{M}$ is a locally product Riemannian manifold and using the same reasoning that in the previous theorem we obtain

$$0 = g(\kappa\mathbf{n}, F\mathbf{t}). \tag{4.4}$$

Then, either $\kappa = 0$ and the curve is a geodesic or $N\mathbf{t}$ is orthogonal to $\mathbf{n}$. If $Nt = 0$ it is an invariant curve, in other cases it is

$$N\mathbf{t} = \pm\sin\theta\mathbf{b}, \tag{4.5}$$

with $\theta \in (0, \pi)$ the slant angle.

Since

$$0 = g(\mathbf{t}, N\mathbf{t}) = g(F\mathbf{t}, FN\mathbf{t}) = g(T\mathbf{t}, FN\mathbf{t}) + g(N\mathbf{t}, FN\mathbf{t}),$$

using (2.5) we get

$$\begin{aligned} g(\mathbf{b}, F\mathbf{b}) = \frac{1}{\sin^2\theta}g(N\mathbf{t}, FN\mathbf{t}) = \frac{-1}{\sin^2\theta}g(T\mathbf{t}, tN\mathbf{t}) = -g(T\mathbf{t}, \mathbf{t}) \\ = -\cos\theta. \end{aligned} \tag{4.6}$$

At every point of the curve $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and $\{F\mathbf{t}, F\mathbf{n}, F\mathbf{b}\}$ are two orthonormal basis. If F preserves the orientation

$$F\mathbf{n} = F\mathbf{b} \times F\mathbf{t} = (g(F\mathbf{b}, \mathbf{t})\mathbf{t} + g(F\mathbf{b}, \mathbf{n})\mathbf{n} - \cos\theta\mathbf{b}) \times F\mathbf{t}.$$

Using this and (4.5) and (4.6):

$$\begin{aligned} g(\mathbf{n}, F\mathbf{n}) &= g(\mathbf{n}, (g(F\mathbf{b}, \mathbf{t})\mathbf{t} + g(F\mathbf{b}, \mathbf{n})\mathbf{n} - \cos\theta\mathbf{b}) \times F\mathbf{t}) \\ &= g(F\mathbf{b}, \mathbf{t})g(\mathbf{n}, \mathbf{t} \times N\mathbf{t}) - \cos\theta g(\mathbf{n}, \mathbf{b} \times T\mathbf{t}) \\ &= -\sin^2\theta - \cos^2\theta = -1. \end{aligned} \tag{4.7}$$

Before computing the torsion we need one more result. As $\mathbf{n}$ is orthogonal to $T\mathbf{t}$, differentiating

$$0 = \mathbf{t}g(\mathbf{n}, T\mathbf{t}) = g(\nabla_{\mathbf{t}}\mathbf{n}, T\mathbf{t}) + g(\mathbf{n}, \nabla_{\mathbf{t}}T\mathbf{t}) = g(\nabla_{\mathbf{t}}\mathbf{n}, T\mathbf{t}) + g(\mathbf{n}, \cos\theta\nabla_{\mathbf{t}}\mathbf{t}). \tag{4.8}$$

Finally, taking into account (4.3) and (4.5):

$$\tau = g(\nabla_{\mathbf{t}}\mathbf{n}, \mathbf{b}) = \pm\frac{1}{\sin\theta}g(\nabla_{\mathbf{t}}\mathbf{n}, N\mathbf{t}) = \pm\frac{1}{\sin\theta}(g(\nabla_{\mathbf{t}}\mathbf{n}, F\mathbf{t}) - g(\nabla_{\mathbf{t}}\mathbf{n}, T\mathbf{t})). \tag{4.9}$$

As $\mathbf{n}$ is normal to $N\mathbf{t}$ and $T\mathbf{t}$ it is normal to $F\mathbf{t}$, then the first addend is

$$g(\nabla_{\mathbf{t}}\mathbf{n}, F\mathbf{t}) = -g(\mathbf{n}, \nabla_{\mathbf{t}}F\mathbf{t}) = -g(\mathbf{n}, F\nabla_{\mathbf{t}}\mathbf{t}) = -g(\mathbf{n}, \kappa F\mathbf{n}) = \kappa,$$

where we have used (4.7). And using (4.8) in the second addend, (4.9) gives $\tau = \pm\frac{\kappa(1+\cos\theta)}{\sin\theta}$.

If F does not keep the orientation, $F\mathbf{b} = -F\mathbf{n} \times F\mathbf{t}$ but we obtain the same result. □

This statement is related with the Lancret Theorem for generalized helices in $\mathbb{R}^3$, which says that the ratio of torsion to curvature is constant. In this sense, a curve which is a slant submanifold is a generalized helix.

5. Curves in the almost product Euclidean plane

From Theorem 4.2 the quotient κ/τ is constant, and the given examples already have constant curvature and torsion. But those were quite straightforward examples, with simple g and F. Are there more examples with nonconstant curvature and torsion? It would be necessary to choose the metric, the structure with $\nabla F = 0$, and look for a curve satisfying the slant condition. It remains as an open problem.

If we remove the locally product condition we can present a huge variety of examples. We will follow a completely different point of view, beginning with a curve and searching for a structure, compatible with the metric, such as it makes the curve to be a slant submanifold.

Theorem 5.1 *For any curve γ in the Euclidian plane and any constant angle θ, we can locally give an almost product structure over the plane such that γ is a slant submanifold with slant angle θ.*

Proof Consider the Euclidean plane $(\mathbb{R}^2, g)$. An almost product structure is given by a matrix $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, for certain functions a, b, c, d. But because of the compatibility with the metric (2.1) it could be written as

$$F = \begin{pmatrix} \cos A(x,y) & \sin A(x,y) \\ \sin A(x,y) & -\cos A(x,y) \end{pmatrix}, \tag{5.1}$$

at any point (x, y), for certain function $A(x, y)$.

Given a curve $\gamma(s)$, if it is not a vertical line, for any point $P = \gamma(s_0) = (x_0, y_0)$, there exists an environment where the curve can be written as a graph $\gamma(x) = (x, y(x))$, $x \in (x_0 - \epsilon, x_0 + \epsilon)$. The tangent vector field is $\gamma'(x) = (1, y'(x))$. Therefore the slant condition reduces to:

$$\begin{aligned} \cos\theta &= \frac{g(\gamma', F\gamma')}{|\gamma'||F\gamma'|} \\ &= \frac{\cos A(x) + 2y'(x)\sin A(x) - y'(x)^2\cos A(x)}{1 + y'(x)^2}, \end{aligned} \tag{5.2}$$

where we have written $A(x) = A(x, y(x))$. We can obtain $A(x)$ as an implicit function at least in an environment $x \in (x_0 - \epsilon_1, x_0 + \epsilon_1)$. Actually, it is just defined at the points of the curve. We could extend the definition for all the points of the vertical strip $(x, y) \in (x_0 - \epsilon, x_0 + \epsilon) \times \mathbb{R}$ as $A(x, y) = A(x)$.

With this almost contact structure, the curve in an environment of P is a slant submanifold with slant angle θ. $\square$

There are two points we want to clarify. First, we use that, at one point of the curve, it can be written as a graph. This has been made just in order to simplify the computations. Also, once the almost product structure is defined over the curve, different extensions could be considered to an environment.

Note that the same process could be done in higher dimensions: choose a curve (with constant/nonconstant curvatures) in $\mathbb{R}^n$, find an almost product structure F over the curve, such as the curve is a slant submanifold with the desired angle and extend the structure to an environment.

We give an illustrative example. Consider $\gamma(x) = (x, e^x)$. Equation (5.2) implies

$$A(x) = \arccos\left(\frac{\cos\theta(1 - e^{2x}) - 2e^x \sin\theta}{1 + e^{2x}}\right),$$

and the almost product structure is obtained from (5.1) with $A(x, y) = A(x)$. Note that, in this case, it is defined in the whole plane.

6. Curves in hypersurfaces of a Kaehler manifold

A hypersurface of a Kaehler manifold can be endowed with a canonical almost product structure [6]. Let $\widetilde{M}^{2n+1}$ be an oriented real hypersurface of a Kaehler manifold (M', J, g) with complex dimension $n+1$, an let V be the unit normal vector field. Consider $\widetilde{M}$ with the induced metric and the following tensor

$$FX = -X + 2g(X, JV)JV. \tag{6.1}$$

this F endows M with an almost product structure.

In the next result, we characterize which curves in the hypersurface are slant submanifolds.

Theorem 6.1 *Let γ be a smooth curve parametrized by arc length of an oriented hypersurface $\widetilde{M}$ of a Kaehler manifold M'. Then, γ is a slant submanifold of $\widetilde{M}$ if and only the tangent vector field $\mathbf{t}$ makes a constant angle with the direction JV.*

Proof Let $\mathbf{t}$ be the tangent vector of the curve. For any curve γ, the angle between $\mathbf{t}$ and $F\mathbf{t}$ is given by

$$\cos\theta = \frac{g(\mathbf{t}, F\mathbf{t})}{|\mathbf{t}||F\mathbf{t}|} = -g(\mathbf{t}, \mathbf{t}) + 2g(\mathbf{t}, JV)^2 = 2g(\mathbf{t}, JV)^2 - 1, \tag{6.2}$$

where we have used the canonical almost product structure (6.1).

This angle is constant, and in such a case γ is a slant submanifold, if and only if $g(\mathbf{t}, JV)$ is constant, which means that the curve makes a constant angle with the direction JV. $\square$

Example 6.2 *Consider* $\mathbb{S}^3$ *in* $\mathbb{C}^2$ *with its Kaehler structure. A curve in* $\mathbb{S}^3$ *is given by*

$$\gamma(t) = (\cos t \cos\psi(t), \cos t \sin\psi(t), \sin t \cos\psi(t), \sin t \sin\psi(t)),$$

for certain function $\psi(t)$.

Imposing

$$\frac{g(\mathbf{t}, JV)}{|\mathbf{t}||JV|} = \cos\alpha$$

leads to

$$\frac{-2\sin\psi(t)\cos\psi(t)\psi'(t)}{\sqrt{1+\psi'(t)^2}} = \cos\alpha.$$

For any solution of this equation, we would obtain a curve with slant angle $\arccos(2\cos\alpha - 1)$.

Example 6.3 *Consider the immersion of* $\mathbb{R}^3 \hookrightarrow \mathbb{C}^2 : (x, y, z) \to (x, y, z, z)$. *In this case* $V = 1/\sqrt{2}(0, 0, -1, 1)$. *Consider a curve*

$$\gamma(t) = (x(t), y(t), z(t), z(t)),$$

parameterized by its archlenght. The condition given by the theorem is

$$\frac{g(\mathbf{t}, JV)}{|\mathbf{t}||JV|} = -\sqrt{2}\dot{z} = \cos\alpha,$$

and therefore:

$$z(t) = -\frac{1}{\sqrt{2}}\cos\alpha t + c.$$

Also, from the unit condition:

$$\dot{x}^2 + \dot{y}^2 + 2\dot{z}^2 = 1,$$

$$\dot{x}^2 + \dot{y}^2 = \sin^2\alpha.$$

We are looking for a particular solution, so we choose

$$\begin{array}{rcl} \dot{x}(t) & = & \sin\alpha\cos(t), \\ \dot{y}(t) & = & \sin\alpha\sin(t). \end{array}$$

We take the initial condition $\gamma(0)$ *as the origin. The equations of the desired curve are*

$$x(t) = \sin\alpha\sin(t), \quad y(t) = -\sin\alpha\cos(t) + \sin\alpha, \quad z(t) = -\frac{1}{\sqrt{2}}\cos\alpha t.$$

Acknowledgment

We want to thank the reviewers for their appropriate suggestions which have improved the paper.

Both authors are partially supported by the PAIDI group FQM-327 (Junta de Andalucía). The second one is a member of the Institute of Mathematics of the University of Seville (IMUS).

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